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Is Quantized Electric Charge a Purely Electromagnetic Phenomenon?

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It is observed that the manifestly covariant Feynman path integral formulation for quantum electromagnetism admits a physically interesting extended definition for the sum-over-histories measure. From an equal-weighting condition and the postulate that the functional integration is to be free of renormalization, it follows that point singularies in the electromagnetic field have an electric charge associated with the fine-structure value $\alpha = (137.032\,41)^{-1}$.

In the manifestly covariant path integral formulation for the quantum theory of electromagnetic radiation, the Feynman sum-over-histories for the probability amplitude is given as[1]

$$K(\sigma_2, \sigma_1) \equiv \int_C e^{iS[A]} D[A] \tag{1}$$

in which (natural units: $\hbar = c = 1$, $e^2 \cong 4\pi/137.036$)

$$S[A] \equiv \int_{\sigma_1}^{\sigma_2} \mathscr{L} d^4x, \qquad \mathscr{L} \equiv -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu}, \qquad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2}$$

Here σ_1 and σ_2 denote flat three-dimensional spacelike hypersurfaces orthogonal to a timelike constant unit vector n_μ ($n^\nu n_\nu = -1$), and C is the class of continuous $A = [A_0(x), A_1(x), A_2(x), A_3(x)]$ through the space-time volume which yield prescribed values for

$$B_{\mu\nu} \equiv -B_{\nu\mu} \equiv F_{\mu\nu} + n_\mu n^\rho F_{\rho\nu} + n_\nu n^\rho F_{\mu\rho} \qquad (B_{\mu\nu} n^\nu \equiv 0) \tag{3}$$

[1] For recent reviews and applications of the Feynman formulation see Papadopoulos and Devreese (1978). Renormalization with displacement-invariant measure is discussed and employed in the present author's paper in this volume, pp. 201–235.

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over σ_1, σ_2. The state of the quantum radiation field is a complex-valued functional of the three independent components in the tensor (3) (i.e., the magnetic field components for $n_\mu = \delta_{\mu 0}$). Displacement invariance of the measure in (1),

$$D[A] \equiv D[A + \delta A] \tag{4}$$

for arbitrary continuous δA that vanish over σ_1, σ_2, insures gauge invariance for (1) and produces the operator field equations

$$\int_C \partial_\mu F^{\mu\nu}(x) e^{iS[A]} D[A] = 0 \tag{5}$$

as well as the commutation relations and time-ordered product formulas for the radiation field. For the explicit evaluation of (1) it is necessary to introduce a space-time lattice $\Lambda(\lambda)$ with point separation distance λ, a lattice generated through the four-volume by translations of a normalized vierbein with n_μ the timelike member. Over this space-time lattice, the action and measure in (1) are represented as

$$S[A] \doteq \sum_{x \in \Lambda(\lambda)} [\mathscr{L}(x; \lambda)\lambda^4], \qquad D[A] \doteq \prod_{x \in \Lambda(\lambda)} \left[\mathscr{N} \prod_{\mu=0}^{3} dA_\mu(x)\right] \tag{6}$$

in which $\mathscr{L}(x; \lambda)$ is the finite-difference correspondent of the Lagrangian in (2), $\mathscr{N} = \mathscr{N}(\lambda)$ is the appropriate normalization constant, and the $\lambda \to 0$ limit is understood to be taken as the final step in the calculation.

To incorporate processes involving electric charge-carrying particles, the correct procedure is to add $A_\mu J^\mu$ (depending on other fields) + (kinematic terms for the other fields) to $\mathscr{L}$, as in quantum electrodynamics. This extension yields an accurate theory for processes involving photons and leptons but affords no explanation for the value of the fundamental unit of charge e. In the unified field theories for the electromagnetic and weak interactions (Weinberg, 1967; Salam, 1968; Georgi, 1974; Hsu, 1976), connection formulas between e and the more primary coupling constants are derived. However, a unified-theory connection formula does not preclude the possibility that the value of e is already fixed in the context of a complete theory for quantum electromagnetism, in which the field itself controls and determines the admissible strength of its point singularities. This is of course an old and aesthetically appealing idea, but we now know that it cannot be realized in Nature by a nonlinear Lagrangian in place of $\mathscr{L}$: the linearity of the operator field equations (5) is in strict accord with the experimental supposition property of electromagnetic radiation, and the accurate theory for processes involving photons and leptons is based precisely on $\mathscr{L}$ and the associated action given in (2).

Because the action in (1) is experimentally ironclad and unalterable, the only thing that may be amenable to an extended definition is the measure. Let us consider a defining condition on the measure somewhat weaker than the unrestricted displacement-invariance condition (4) of radiation theory. Specifically, let (4) hold only for $\delta A(x) = \hat{\mathscr{L}}(x)\,\delta\xi(x)$, where $\delta\xi_\mu(x)$ denotes an arbitrary continuous infinitesimal four-vector that vanishes over σ_1, σ_2 and where $\hat{\mathscr{L}}(x) = \lim_{\lambda\to 0}\hat{\mathscr{L}}(x;\lambda)$ is computed from the field history A in (4) by

$$\hat{\mathscr{L}}(x;\lambda) \equiv \mathscr{L}(x;\lambda) - \tfrac{1}{2}\sum_{i,j,k,l}{}' A^\mu(x+\lambda u_{(i)})A_\mu(x+\lambda u_{(j)})A^\nu(x+\lambda u_{(k)})A_\nu(x+\lambda u_{(l)}) \tag{7}$$

Restricted to x and its nearest spacelike neighboring points in the lattice, the summation in (7) involves 7 four-vectors: $u_{(0)} = (0,0,0,0)$ and $u_{(i)}$ for $i = \pm 1, \pm 2, \pm 3$ defined by the conditions $n_\mu u^\mu_{(i)} = 0$, $u_{(-i)} = -u_{(i)}$, $u_{(i)\mu}u^\mu_{(j)} = \delta_{ij}$ [which give $u^\mu_{(i)} = (\operatorname{sgn} i)\delta_i{}^\mu$ for $n_\mu = \delta_{0\mu}$]. The $\sum'$ summation in (7) is required in order to exclude third and fourth powers of any component of A at a lattice point, and also terms with the dominating structure $A^\mu(x)A_\mu(y)A^\nu(x)A_\nu(y)$ for a pair of lattice points; the latter terms must not appear in (7) for a realizable constraint on the measure without renormalization (Friedricks and Shapiro, 1957), because $\mathscr{L}(x;\lambda)$ is composed exclusively of terms bilinear and square in components of A at lattice points. Thus, by definition, with the $i = j$, $k = l$, and $(i,j) = (k,l)$ terms excluded, there are $7\cdot 6(7\cdot 6 - 1) = 1722 \equiv N'$ terms in the $\sum'$ summation. In the case of a field history such that $\hat{\mathscr{L}}(x) = 0$ at a certain x, we have admissible $\delta A(x) = \hat{\mathscr{L}}(x)\,\delta\xi(x) = 0$ and (4) is satisfied trivially for $\delta\xi(x)$ concentrated at the point x. The measure is as given in (6) but with $\mathscr{N} \equiv \bar{\mathscr{N}}(\lambda) + \bar{\bar{\mathscr{N}}}(\lambda) \times$ [Dirac δ function of $\hat{\mathscr{L}}(x;\lambda)$].

Now by making a replacement of the dummy integration variable in (1) $A \to A + \delta A$ and subtracting the resulting equation from (1), we have

$$\int_C \left(e^{iS[A]}D[A] - e^{iS[A+\delta A]}D[A+\delta A]\right) = 0 \tag{8}$$

for arbitrary δA. Evoking (4) specifically for $\delta A = \hat{\mathscr{L}}\delta\xi$ with $\delta\xi$ arbitrary through the interior of the four-volume, (8) yields the operator field equations

$$\int_C \hat{\mathscr{L}}(x)\partial_\mu F^{\mu\nu}(x)e^{iS[A]}D[A] = 0 \tag{9}$$

The latter are the quantum correspondent of the classical field equations

$$\hat{\mathscr{L}}(x)\partial_\mu F^{\mu\nu}(x) = 0 \tag{10}$$

which follow from (2) and the variational principle $\delta S[A] \equiv S[A+\delta A] - S[A] = 0$ for $\delta A = \hat{\mathscr{L}}\delta\xi$. Equation (10) admits charge-current point singularities with $\partial_\mu F^{\mu\nu}(x)$ proportional to a three-dimensional δ function in the spatial coordinates only if $\mathscr{L}(x) = 0$ at the point. Thus, by (7), the condition

$$\tfrac{1}{2}F^{\mu\nu}F_{\mu\nu} + N'(A^\mu A_\mu)^2 = 0 \tag{11}$$

is required to hold at a point singularity. By the usual association, field histories with point singularities contribute constructively to the functional integration in (1) only if the field history satisfies (11) at the singularity points.

Of the possible relations between A_μ and $F_{\mu\nu}$ at a point singularity, (11) (with N' fixed appropriately on the classical level) is unique in that it alone holds for a point charge moving in an unrestricted way with arbitrary acceleration (Schiff, 1969). This is immediately verified by substituting the Wiechert potentials for a point charge in arbitrary motion into (11), a test which other possible relations between A_μ and $F_{\mu\nu}$ fail. Condition (11) is also invariant under the 15-parameter group of conformal coordinate transformation (McLennan, 1957; Rosen, 1972). The value $N' = 1722$ emerges here with equal weight given to the terms in (7) and for the $\sum'$ summation that keeps the functional integration over A well defined without renormalization. By substituting the general Wiechert potentials into (11) (or more simply, the specialized solution $A_\mu = (e/4\pi r)\delta_{\mu 0}$ for a point-charge at rest at the origin of the spatial coordinates), we obtain the electric charge values $e = \pm 4\pi/(N')^{1/2} = \pm 0.302\,826\,09$ and the fine-structure constant value $\alpha = e^2/4\pi = 4\pi/N' = (137.032\,41)^{-1}$. The small difference between the latter theoretical and the current experimental value may be due to a calculable radiative effect.[2]

This theory for quantum electromagnetism does not impose any constraint on the motion of its point charges. To describe the motion of the point charges, additional terms must be added to $\mathscr{L}$ in the usual manner.

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[2] At the present time we have $\alpha_{\mathrm{exp}} = (4\pi/N')[1 - \tfrac{1}{2}\alpha^2 + 0(\alpha^3)]$.